Premium Calculation Based on Physical Principles

Amir H. Darooneh

Department of Physics, Zanjan University,

*P.O.Box 45196-313, Zanjan, Iran.**

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Abstract

We consider the concept of equilibrium in economic systems from statistical mechanics viewpoint. A new method is suggested for computing the premium on this basis. The Bühlmann economic premium principle is derived as a special case of our method.

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*Electronic address: darooneh@mail.znu.ac.ir

I. INTRODUCTION

Methods developed in physics are widely used for modeling and data analysis of the financial market. This approach to quantitative economy was adopted by economists in nineteen century and the beginning of the twentieth century, however it was only during the last decade that physicists turned their attention to problems of such nature [1]. Among all branches of physics, the statistical mechanics appears as the most suitable context for studying the dynamics of complex systems such as financial market [2, 3]. There are many cases which demonstrate the power of statistical mechanics in exploring dynamics of the financial market. Recently Sornette and Zhou [4] show the advantage of applying the principles of statistical mechanics to price prediction in the US market. The study of insurance with the aid of ideas borrowed from statistical mechanics was begun by the work of author [5, 6, 7, 8]. In this paper we try to explain how this approach may be used for pricing the insurance. In what follows we first describe the equilibrium in the economic systems from physical viewpoint then we suggest a new way for calculation of the insurance premium. Like the economic model of Buhlmann [9, 10] the role of market is taken into the account when premium is assigned to a category of risks. The Esscher principle also appears as a special case of this method.

II. ECONOMIC SYSTEMS IN EQUILIBRIUM

The financial market is combination of large number of economic agents which are interacting with each other through buying and selling. We consider the behavior of one of the agents for example an insurance company; all other agents may be regarded as its environment. The agent exchanges money when interacts with its environment. We suppose the financial market is a closed system and the clearing condition is satisfied. This means, the environment absorbs the money that the agent loses and will supply the agent's incomes.

$$w_a + w_e = W_m = Const. \tag{1}$$

The quantities w_a,w_e and W_m are the wealth of agent, its environment and total money in the market respectively.

In a given period of time, there are many ways for the agent to possess amounts of money as a result of random loses and incomes in its trading. The quantity $\Gamma_a(w_a)$ represents

number of these ways. The environment has also $\Gamma_e(w_e)$ ways to acquire amounts of money as its wealth. The state of the market is specified by two quantities w_a andw_e, the market has $\Gamma_m(w_a, w_e)$ ways of reaching this specified state. Clearly,

$$\Gamma_m(w_a, w_e) = \Gamma_a(w_a)\Gamma_e(w_e). \tag{2}$$

Our common sense tells us, at any time the market chooses any one of these ways with equal probability because no reason exists for preferring some of them. By definition, in equilibrium state the agent and its environment have most options for buying or selling. Existence of any restriction disturbs the equilibrium and decreases the number of ways that may be chosen for trading. Mathematically this means in the equilibrium state the function $\Gamma_m(w_a, w_e)$ should be maximized [11].

$$\frac{\partial \Gamma_m(w_a, w_e)}{\partial w_a} = 0. \tag{3}$$

The agent's wealth in the equilibrium state, W_a, is obtained by solving the above equation. Combination of the eqs. 3 and 2 lead us to the following equality,

$$\left(\frac{\partial \Gamma_a(w_a)}{\partial w_a}\right)_{w_a=W_a} \Gamma_e(W_e) + \Gamma_a(W_a)\left(\frac{\partial \Gamma_e(w_e)}{\partial w_e}\right)_{w_e=W_e} . \frac{\partial w_e}{\partial w_a} = 0. \tag{4}$$

Where W_e represents the equilibrium value for the environment's wealth. By the aid of eq.1 we have $\partial w_e/\partial w_a = -1$, then the equilibrium condition is reduced to its new form.

$$\left(\frac{\partial \ln \Gamma_a(w_a)}{\partial w_a}\right)_{w_a=W_a} = \left(\frac{\partial \ln \Gamma_e(w_e)}{\partial w_e}\right)_{w_e=W_e}. \tag{5}$$

The value of parameter $\partial \ln \Gamma(w)/\partial w$ is denoted by the symbol β, whence the condition for equilibrium becomes,

$$\beta_a = \beta_e. \tag{6}$$

The above equality is useless unless the parameter $\beta_a(\beta_e)$ can be expressed in terms of measurable quantities of the agent (environment). For an insurance company we express this parameter in terms of initial wealth of company, mean claim size and ultimate ruin probability.

III. THE CANONICAL ENSEMBLE THEORY IN ECONOMICS

What is the probability that the agent possesses the specified amount $W_a^{(r)}$ when it is in equilibrium with its environment? From basic probability theory we know it should be

directly proportional to the number of possible ways that are related to this state of the market.

$$Pr(W_a^{(r)}) \propto \Gamma_m(W_a^{(r)}, W_e^{(r)}) = \Gamma_a(W_a^{(r)})\Gamma_e(, W_e^{(r)}). \tag{7}$$

The environment is supposed to have much money in comparison to the agent's wealth,

$$\frac{W_a^{(r)}}{W_m} \ll 1. \tag{8}$$

It is clear that $\Gamma_e(, W_e^{(r)})$ is also much larger than $\Gamma_a(W_a^{(r)})$ hence the eq. 7 can be approximated as,

$$Pr(W_a^{(r)}) \approx \Gamma_e(W_e^{(r)}) = \Gamma_e(W_m - W_a^{(r)}). \tag{9}$$

We can expand the logarithm of above equation around the value .

$$\begin{aligned}
\ln Pr(W_a^{(r)}) &= \ln \Gamma_e(W_m - W_a^{(r)}) \\
&= \ln \Gamma_e(W_m) + \left(\frac{\partial \ln \Gamma_e(w)}{\partial w}\right)_{w=W_m} \left(-W_a^{(r)}\right) + \cdots \\
&\approx \ln \Gamma_e(W_m) - \beta W_a^{(r)}.
\end{aligned} \tag{10}$$

The first term in right hand side of the eq.10 is a constant number and the second term is nothing except of product of parameter β and the agent's wealth. The eq. 8 insures that other terms in the above expansion are small with respect to these leading terms. By a simple algebraic manipulation we obtain the desired result.

$$Pr(W_a^{(r)}) = \frac{e^{-\beta W_a^{(r)}}}{\sum_r e^{-\beta W_a^{(r)}}}. \tag{11}$$

All the accessible equilibrium states of the market make an ensemble of possible values for the agent's wealth. The index r indicates members of this ensemble. This is what the physicists called canonical ensemble.

IV. THE INSURANCE PRICING

The insurance is a contract between insurer and insurant. Any happening loss incurred on the insurant party is covered by insurer, in return for amount of money received as premium.

The wealth of insurer at the end of time interval is,

$$W(t) = W_0 + S(t). \tag{12}$$

Where W_0 and $S(t)$ are initial wealth and surplus of the insurer respectively. The surplus of the insurer is the sum of all loss events and incomes corresponding to each category of the insurance contracts.

$$\begin{aligned} S(t) &= \sum_{\alpha} S_\alpha(t) \\ &= \sum_{\alpha} (p_\alpha I_\alpha(t) - \sum_{j=1}^{N_\alpha(t)} X_j). \end{aligned} \tag{13}$$

The summation goes over different categories. $I_\alpha(t)$ and $N_\alpha(t)$ are number of the insurants and loss events for a specified category respectively. An insurant charges the insurer for X_j in a loss event and pays a premium of p_α to him/her. According to eq. 11, the probability for acquiring the surplus $S(t)$ by the insurer is,

$$Pr(S(t)) = \frac{e^{-\beta S(t)}}{\sum e^{-\beta S(t)}}. \tag{14}$$

The summation goes over all possible values for surplus. The parameter β is positive to ensure that extreme values for surplus have small probability. The number of insurants in eq. 12 is also a random variable and indicates the competition in the market. It may be decreased due to an increment in the premium and will be increased when the insurer reduces the prices. In the case of constant number of insurants we obtain the result similar to what the Buhlmann [9, 10] arrived at in his economic model for insurance pricing.

$$Pr(Z(t)) = \frac{e^{\beta Z(t)}}{\sum e^{-\beta Z(t)}}. \tag{15}$$

The $Z(t)$ demonstrates the aggregate loss in the specified time interval. The insurer naturally aims at maximizing its profit, hence its surplus, after the insurance contract is over, should be positive or zero at least. This condition may be expressed mathematically only as an average form.

$$< S(t) >= \frac{\sum S(t) e^{-\beta S(t)}}{\sum e^{-\beta S(t)}} = 0. \tag{16}$$

When there is no correlation between financial events (loss and gain) in different categories, the eq. 16 is reduced to [9],

$$< S_1(t) >= \frac{\sum S_1(t) e^{-\beta S(t)}}{\sum e^{-\beta S(t)}} = \frac{\sum S_1(t) e^{-\beta S_1(t)}}{\sum e^{-\beta S_1(t)}} = 0. \tag{17}$$

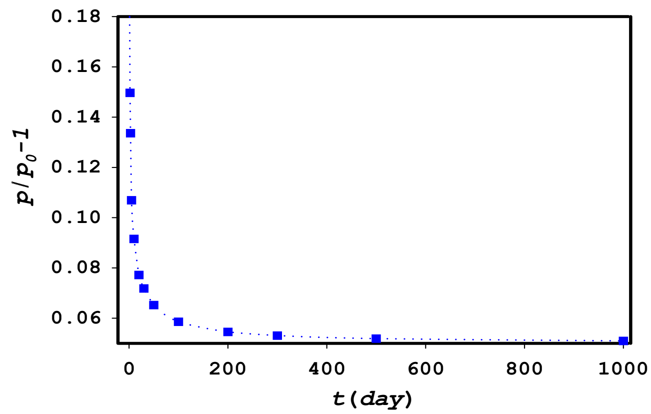

FIG. 1: The loading parameter versus the contract's duration for large β parameter.

In the above equation sum over all equilibrium states is understood. The eq. 17 can be used to compute the premium for corresponding category. For constant number of insurants it is nothing except the Esscher formula for premium assignment to a category of risks.

$$p = \frac{\sum Z_1(t)e^{-\beta Z_1(t)}}{\sum e^{-\beta Z_1(t)}}. \tag{18}$$

So far the parameter β has remained, however by a simple dimensional analysis one can establish that it must be proportional to the inverse of the contracts duration. The eqs. 17 or 18 also shows it is related to premium. The following equation relates ultimate ruin probability to the premium [12].

$$\varepsilon = 1 - \sum_{k=1}^{\infty}(1-\frac{p_0}{p})(\frac{p_0}{p})^k F_e^{*k}(W_0). \tag{19}$$

Where p_0 is the net premium. The $F_e(x)$ is related to claim size probability function $F(x)$,

$$F_e(x) = \frac{1}{\mu}\int_0^x F(y)dy. \tag{20}$$

And $F_e^{*k}(W_0)$ is the k-fold convolution of $F_e(W_0)$ with itself. The parameter μ represents the mean claim size. The ruin probability, initial wealth and mean claim size tune the financial work of insurer and are concealed in the parameter. Combination of the eqs. 17 and 19 shows this fact.

Below, the simulation results for a special case of car insurance are presented for demonstrating some aspects of our method. For simplicity we assume that all cars are the same and pay the same premium, the loss events have exponential distribution for time interval between their occurrence and their size. The necessary data are adopted from reports of

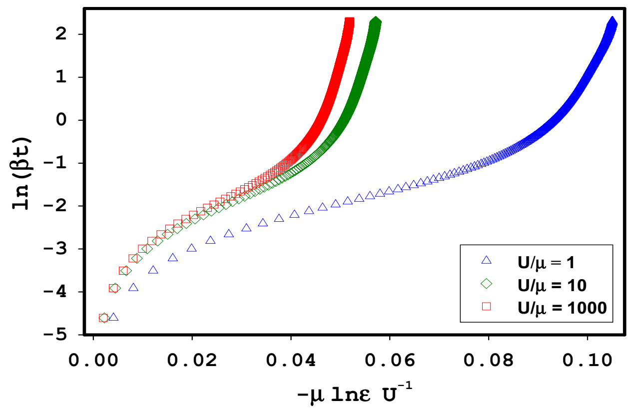

FIG. 2: Dependence of the β parameter on the ruin probability, initial wealth and mean claim size.

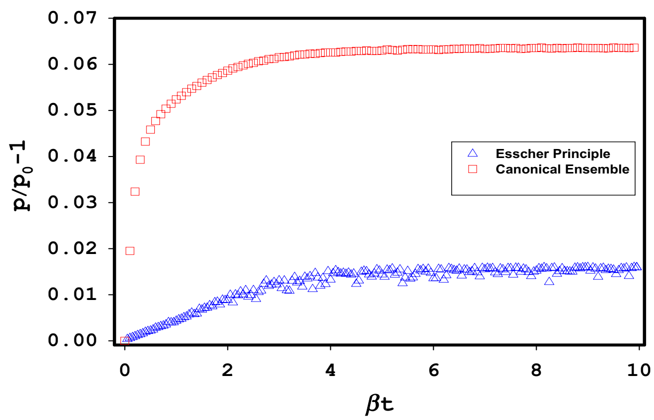

FIG. 3: The loading parameter versus the βt parameter. The squares display the results of the canonical ensemble theory and triangles correspond to Esscher principle.

Iran Central Insurance Company [13]. Fig. 1 demonstrates dependence of loading parameter on the contract's time. It is what we expected in real cases. As we already mentioned, if we specify the ruin probability, initial wealth and mean claim size the β parameter will be determined. Fig. 2 displays the relation between these parameters. The initial wealth is measured in terms of the mean claim size in order to get ride of dependence of the results on the monetary unit. The variation in number of the insurants influences the premium. Fig.3 is plotted to show this effect, the value of the premium which is obtained from Esscher formula is less than that we found in our method.

V. CONCLUSION

Statistical mechanics are pervasively used for studying the financial behavior of an economic agent in the market. The concept of equilibrium is revised in this respect. We prescribe it for the insurance subject and suggest a new way for computing the premium. The effects of interaction between insurer and the market are included in this new method. Some consequences of Bühlmann economic principle are recovered. This work has been supported by the Zanjan university research program on Non-Life Insurance Pricing No: 8243.

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[1] Z. Burda, J. Jurkiewicz and M. A. Nowak, *Is Econophysics a Solid Science?*, arXive:cond-mat/0301096.

[2] J.P. Bouchaud and M. Potters, *Theory of Financial Risks*, Cambridge University Press, Cambridge, (2000).

[3] R. Mantegna and H.E. Stanely, *An Introduction to Econophysics*, Cambridge University Press, Cambridge, (2000).

[4] D. Sornette and W. X. Zhou, *The US 2000-2002 market descent*, Quant. Finance, **2**, 468 (2002).

[5] M. E. Fouladvand and A. H. Darooneh, *Premium Forecasting for an Insurance Company*, Proc. of 2-nd Nikkei Econophys. Symp. 303 (2003).

[6] A. H. Darooneh, *Non Life Insurance Pricing*, Proc. 18-th Ann. Iranian Phys. Conf., 162 (2003).

[7] A. H. Darooneh, *Non Life Insurance Pricing: Statistical Mechanics Viewpoint*, arXiv:cond-mat/0305062.

[8] A. H. Darooneh, *Physics of Insurance*. Will be published in Proc. First Int. Phys. Conf. (2004).

[9] H. Buhlmann, *An Economic Premium Principle*, ASTIN Bulletin,**11**, 52 (1980).

[10] H. Buhlmann, *General Economic Premium Principle*, ASTIN Bulletin,**14**, 13 (1984).

[11] R. K. Pathria, *Statistical Mechanics*, Pergamon Press, Oxford, (1972).

[12] S. A. Klugman, H. H. Panjer and G. E. Willmot, *Loss Models: From Data to Decisions*, John Wiley, New York, (1998).

[13] —— , *Statistical Report of Iranian Insurance Industry*, Iran Central Insurance Company, Tehran, (2000).